Subsets of Singular Cardinals under the Stars

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Col(ω, $< \lambda$)-generic

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How far can we push this analogy?

$I_0(\kappa)$: There is an elementary embedding from $L(\mathcal{P}(\kappa))$ to itself with critical point below κ.

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• ω_2 is measurable and $\mathrm{cof}(\omega_3) = \omega_2$	• Is κ^{++} measurable? $\mathrm{cof}(\kappa^{+++}) = \kappa^{++}$?

Davis (1964): Under $\mathsf{AD}^{L(\mathbb{R})}$, every set of reals in $L(\mathbb{R})$ has the PSP.

Cramer-Shi-Woodin (2015): Under $I_0(\kappa)$, every $A \subseteq \mathcal{P}(\kappa)$ in $L(\mathcal{P}(\kappa))$ has the κ-PSP.

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The model is a Lévy collapse extension.

$$\mathrm{Col}(\omega, < \lambda)$$

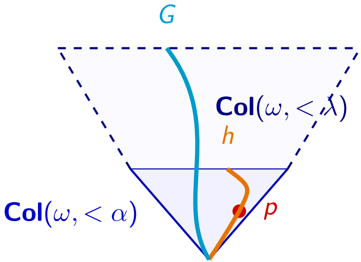
The model is a Merimovich extension.

$$\mathrm{Mer}(\kappa, < \lambda)$$

Lévy collapse $\mathrm{Col}(\omega, <\lambda)$	**Merimovich forcing** $\mathrm{Mer}(\kappa, <\lambda)$
For $\alpha \in (\omega, \lambda)$, it collapses α to ω	For $\alpha \in (\kappa, \lambda)$, it collapses α to κ
It preserves λ	It preserves λ
For $\alpha < \beta \leq \lambda$, $\mathrm{Col}(\omega, <\beta)$ projects into $\mathrm{Col}(\omega, <\alpha)$	For $\alpha < \beta \leq \lambda$, $\mathrm{Mer}(\kappa, <\beta)$ projects into $\mathrm{Mer}(\kappa, <\alpha)$
For each $x \subseteq \omega$ there is $\alpha < \lambda$ such that $x \in V[G \cap \mathrm{Col}(\omega, <\alpha)]$	For each $x \subseteq \kappa$ there is is $\alpha < \lambda$ such that $x \in V[G \cap \mathrm{Mer}(\kappa, <\alpha)]$
For each $\alpha < \lambda$, $\mathrm{Col}(\omega, <\alpha) \in H_\lambda$	For each $\alpha < \lambda$, $\mathrm{Mer}(\kappa, <\alpha) \in H_\lambda$

Interpolation for Lévy collapse $\mathrm{Col}(\omega, < \lambda)$

Let G be $\mathrm{Col}(\omega, < \lambda)$-generic. For every $\alpha < \lambda$ and for every $p \in \mathrm{Col}(\omega, < \alpha)$, there is a $\mathrm{Col}(\omega, < \alpha)$-generic h in $V[G]$ with $p \in h$.



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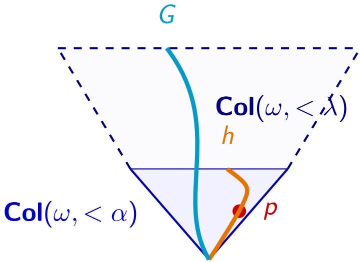

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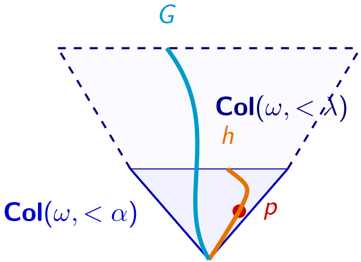

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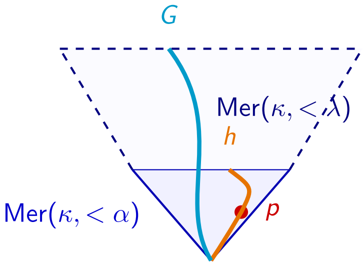

Constellation for Lévy collapse $\text{Col}(\omega, < \lambda)$

Let G be $\text{Col}(\omega, < \lambda)$-generic, and let $\alpha < \lambda$. For each $\text{Col}(\omega, < \alpha)$-generic $h \in V[G]$,

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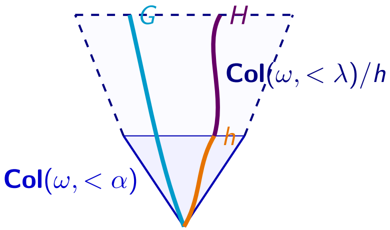
Let G be $\mathrm{Col}(\omega, < \lambda)$-generic, and let $\alpha < \lambda$. For each $\mathrm{Col}(\omega, < \alpha)$-generic $h \in V[G]$, there is a $\mathrm{Col}(\omega, < \lambda)/h$-generic H such that $V[h][H] = V[G]$.



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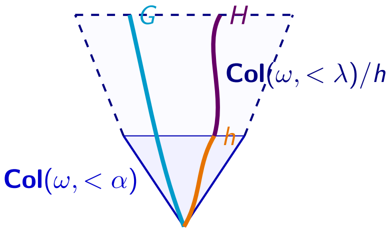

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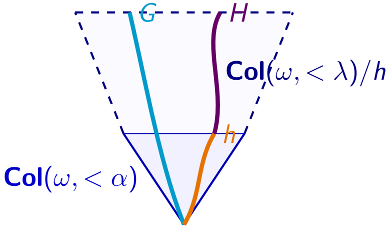

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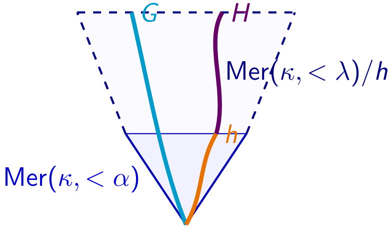
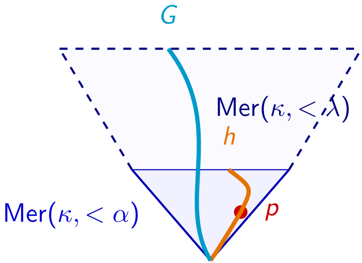


Interpolation



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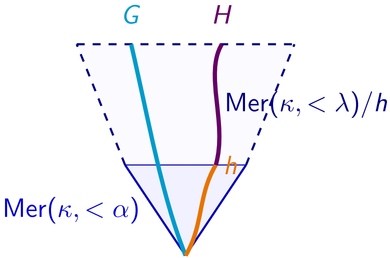

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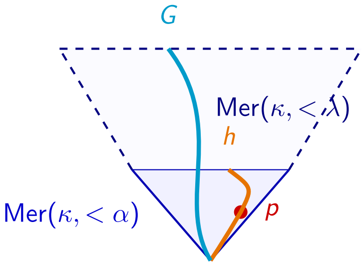

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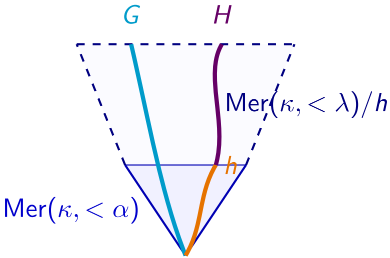
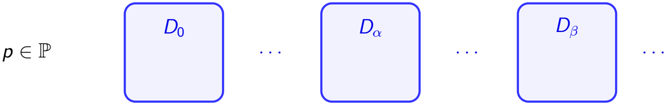


Which forcings satisfy Interpolation and Constellation?

Predistributivity

Definition

A forcing $\mathbb{P}$ is (ω, κ)-*predistributive* if for all $p \in \mathbb{P}$ and $\langle D_\alpha \mid \alpha < \gamma \rangle \in {}^{<\kappa}\mathcal{D}(\mathbb{P})$,



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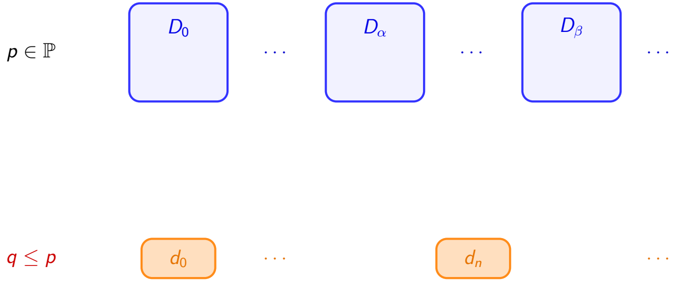

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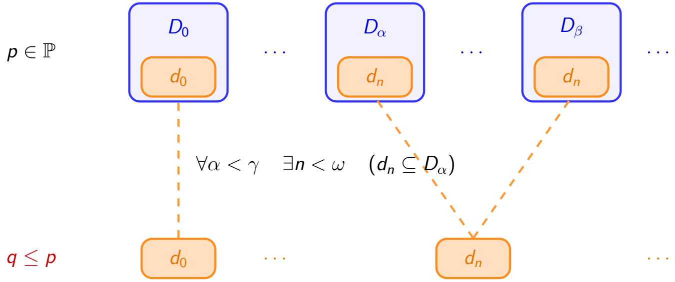

Propaganda

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4. Suppose κ is strongly compact.
   A forcing $\mathbb{Q}$ is (ω, κ)-predistributive if and only if for some $\lambda > \kappa$ and some fine measure $\mathcal{U}$ on $\mathcal{P}_\kappa(\lambda)$, strongly compact Prikry forcing $\mathbb{P}_\mathcal{U}$ projects into RO($\mathbb{Q}$).

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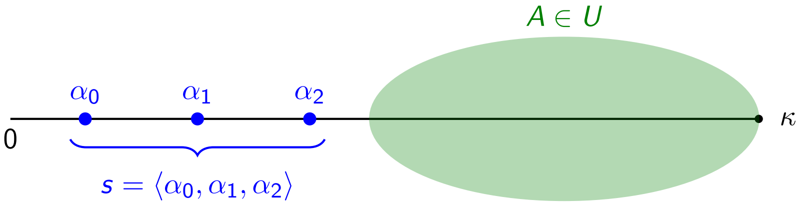
From Prikry forcing $\mathbb{P}_U$ to predistributivity



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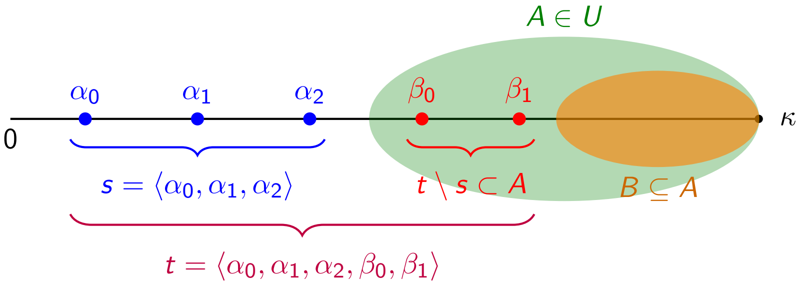
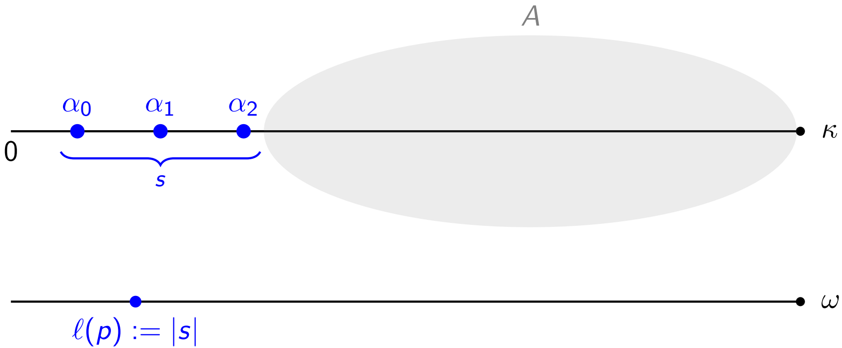


Predist.: $\forall \vec{\mathcal{D}} \in {}^{<\kappa}\mathcal{D}(\mathbb{P})\ \forall p\ \exists q \leq p\ \exists \vec{E} \in {}^{\omega}\mathcal{D}(\mathbb{P}_{\downarrow q})$ such that $\forall \alpha\ \exists n\ \vec{E}_n \subseteq \vec{D}_\alpha$

Suppose $p = (s, A) \in \mathbb{P}_U$ and $\langle D_\alpha \mid \alpha < \gamma \rangle \in {}^{<\kappa}\mathcal{D}(\mathbb{P}_U)$.



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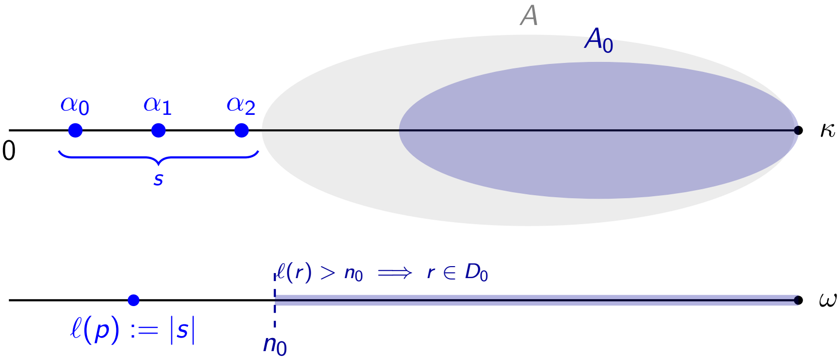
By the Strong Prikry Property, there are $q_0 = (s, A_0) \leq^* p$ and $n_0 < \omega$ such that $\forall r \leq q_0 \big(\ell(r) > n_0 \implies r \in D_0\big)$



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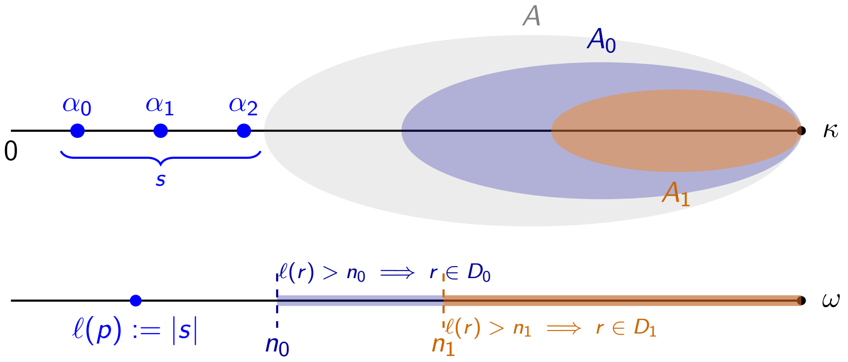
Similarly, there are $q_1 = (s, A_1) \leq^* p$ and $n_1 < \omega$ such that $\forall r \leq q_1 \big(\ell(r) > n_1 \implies r \in D_1\big)$



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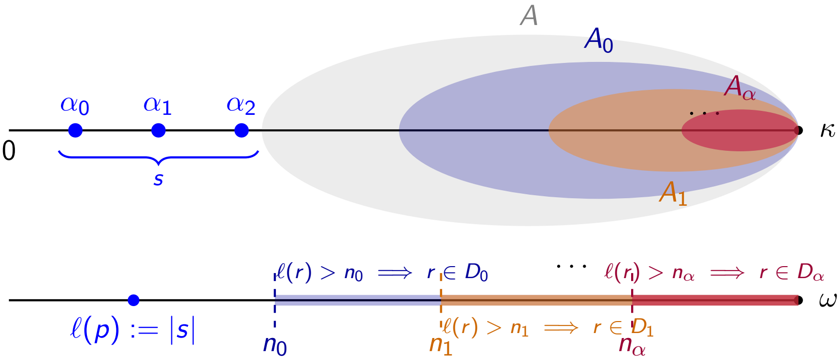
For $\alpha < \gamma$, there are $q_\alpha = (s, A_\alpha) \leq^* p$ and $n_\alpha < \omega$ such that
$\forall r \leq q_\alpha \big(\ell(r) > n_\alpha \implies r \in D_\alpha\big)$



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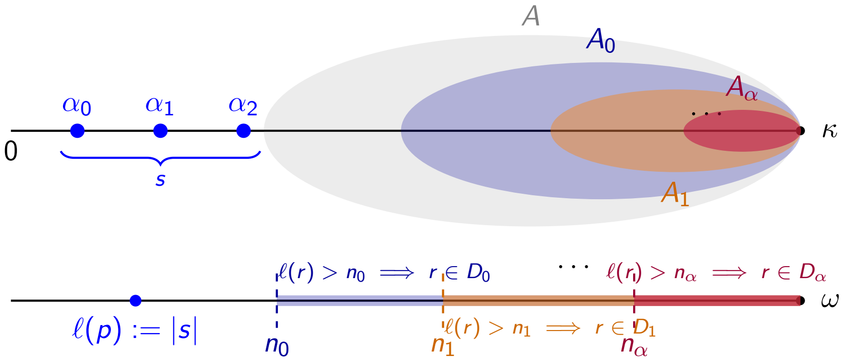
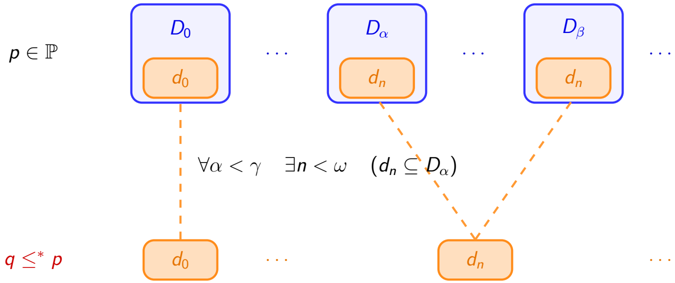
By κ-completeness $\bigcap_{\alpha<\gamma} A_\alpha \in U$, so $q_\infty := (s, \bigcap_{\alpha<\gamma} A_\alpha) \leq^* q_\alpha$, for all $\alpha < \gamma$. Moreover, $\forall \alpha < \gamma \exists m \forall r \leq q_\infty (\ell(r) > m \implies r \in D_\alpha)$.



Vanilla Prikry forcing $\mathbb{P}_U$ is (ω, κ)-predistributive: If $p \in \mathbb{P}_U$ and $\langle D_\alpha \mid \alpha < \gamma \rangle \in {}^{<\kappa}\mathcal{D}(\mathbb{P}_U)$, then there is $q \leq^* p$ such that for all $\alpha < \gamma$ there is $n < \omega$ with $d_n := \{r \leq q \mid \ell(r) > n\} \subseteq D_\alpha$.



A further assumption

- (•) Suppose $V \subseteq M$ are models of ZFC, and let $\langle D_\alpha \mid \alpha < \kappa \rangle \in M$ is an enumeration of $\mathcal{D}(\mathbb{P}_U)^V$.

  - (•) Suppose there is $\langle B_n \mid n < \omega \rangle \in ({}^\omega \mathcal{P}_{\mathrm{bd}}(\kappa))^M$ with $\bigcup_{n<\omega} B_n = \kappa$ and $\langle D_\alpha \mid \alpha \in B_n \rangle \in V$, for all $n < \omega$.

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p_0 

$$\forall \alpha \in B_0 \exists m \forall r \leq p_0 (\ell(r) > m \implies r \in D_\alpha)$$

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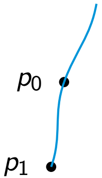

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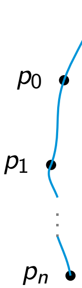

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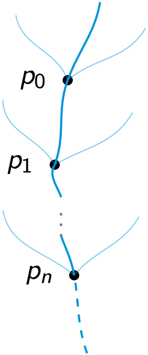
$\mathbb{P}_U$-generic filter in M



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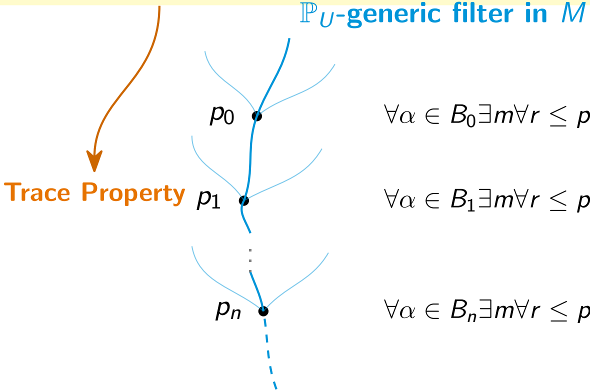

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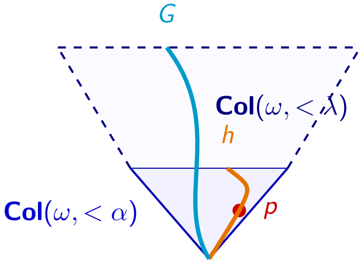
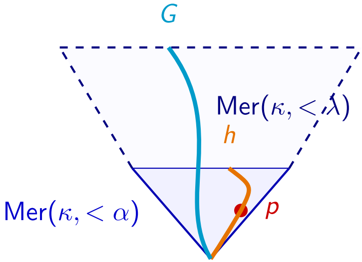
Definition

$M \supseteq V$ has the *κ-trace property* if for all surjections $f\colon \kappa \to E$ in M with $E \in V$, there is a sequence $\langle B_n \mid n < \omega \rangle \in M$ of bounded subsets of κ such that $\bigcup_{n<\omega} B_n = \kappa$ and $f \restriction B_n \in V$, for all $n < \omega$.

Predist.: $\forall \vec{\mathcal{D}} \in {}^{<\kappa}\mathcal{D}(\mathbb{P})\ \forall p\ \exists q \leq p\ \exists \vec{E} \in {}^{\omega}\mathcal{D}(\mathbb{P}_{\downarrow q})$ such that $\forall \alpha\ \exists n\ \vec{E}_n \subseteq \vec{D}_\alpha$
Trace prop.: $\forall f \in {}^{\kappa}E \cap M\ \exists \mathcal{B} \in ({}^{\omega}\mathcal{P}_{\mathsf{bd}}(\kappa))^M$ such that $\kappa = \bigcup \mathcal{B}$ and $f \upharpoonright \mathcal{B}_n \in V$

With Lévy and Merimovich one can construct generic filters for small forcings.





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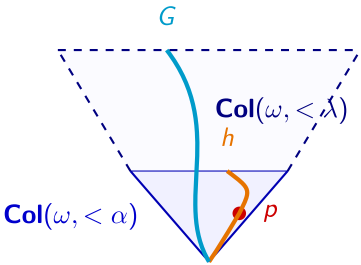

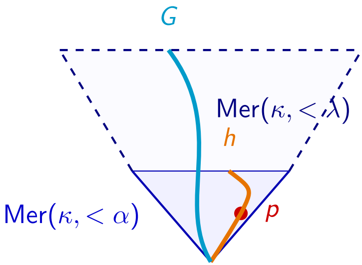

Can one do the same with (ω, κ)-predistributive forcings?

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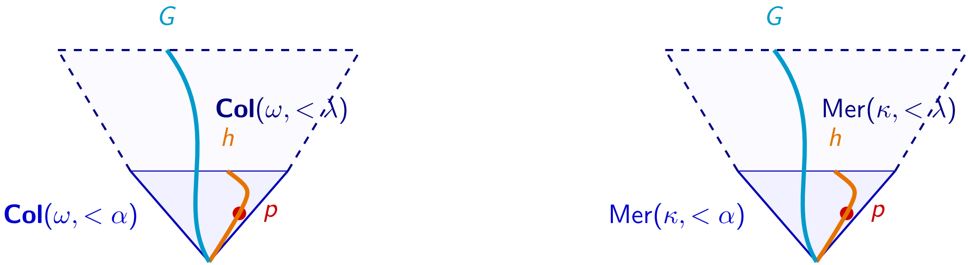

Can one do the same with (ω, κ)-predistributive forcings?
Benhamou-T.-Weltsch: Yes!

Predist.: $\forall \vec{\mathcal{D}} \in {}^{<\kappa}\mathcal{D}(\mathbb{P})\ \forall p\ \exists q \leq p\ \exists \vec{E} \in {}^{\omega}\mathcal{D}(\mathbb{P}_{\downarrow q})$ such that $\forall \alpha\ \exists n\ \vec{E}_n \subseteq \vec{D}_\alpha$
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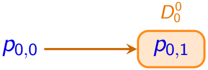
Theorem (Benhamou-T.-Weltsch)

If $M \supseteq V$ has the κ-trace property and $\mathbb{P} \in V$ is (ω, κ)-predistributive, then for all $\langle D_\alpha \mid \alpha < \kappa \rangle \in M$, with $D_\alpha \in \mathcal{D}(\mathbb{P})^V$, there is a filter $g \subseteq \mathbb{P}$ such that $g \cap D_\alpha \neq \emptyset$ for all $\alpha < \kappa$.

Proof. Let $\langle D_\alpha \mid \alpha < \kappa \rangle \in M$ with $D_\alpha \in \mathcal{D}(\mathbb{P})^V$. The κ-trace property yields $\langle B_n \mid n < \omega \rangle \in ({}^\omega\mathcal{P}_{\mathrm{bd}}(\kappa))^M$ such that $\bigcup_{n<\omega} B_n = \kappa$, and $\langle D_\alpha \mid \alpha \in B_n \rangle \in V$.

$\langle D_\alpha \rangle_{\alpha \in B_0} \rightsquigarrow \langle D^0_m \rangle_{m<\omega}$



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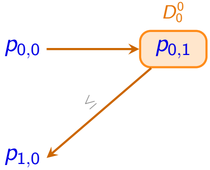

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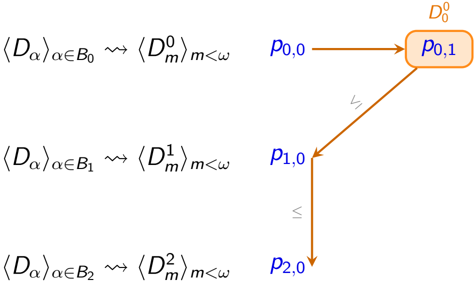

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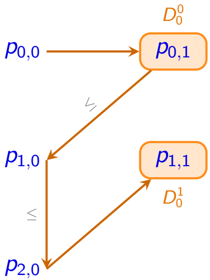

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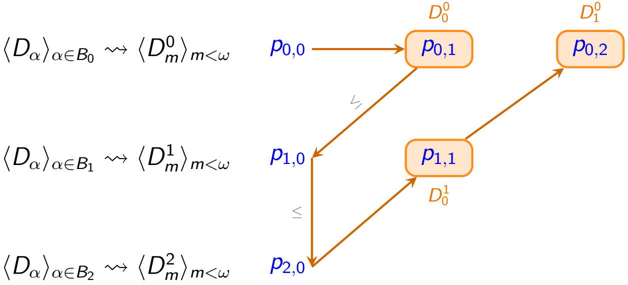

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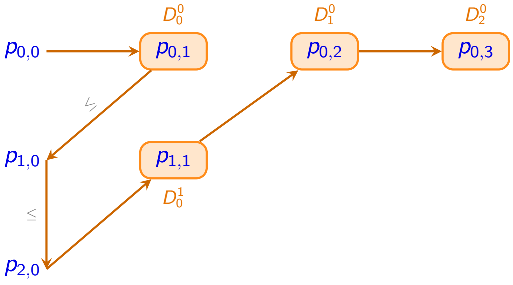

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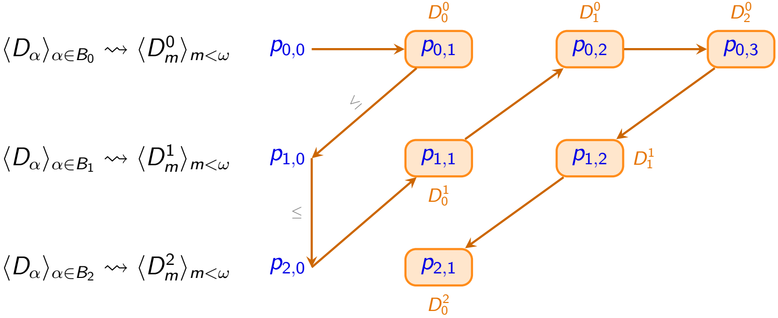

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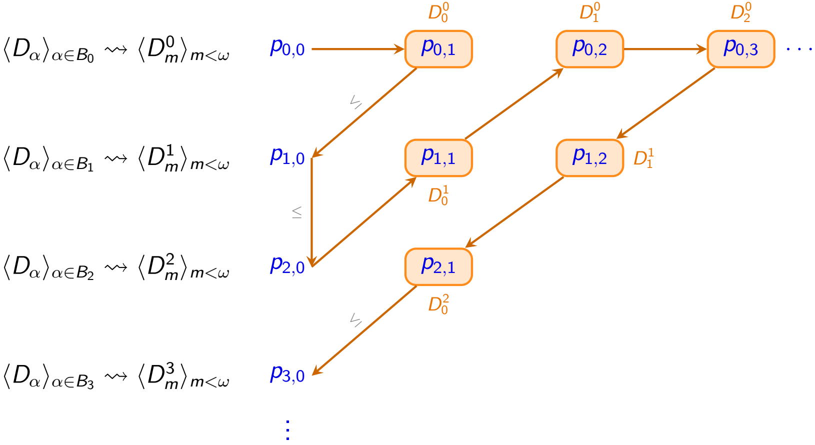

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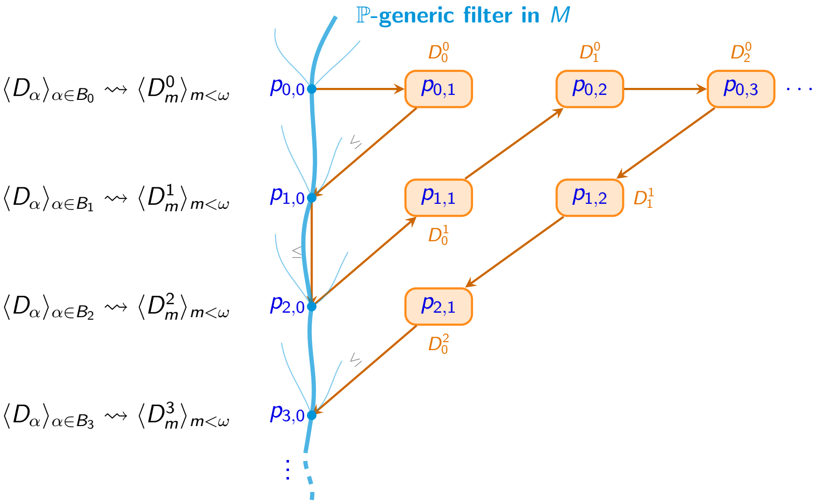

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Corollary

Suppose that

- $M \supseteq V$ has the κ-trace property;
- $\lambda > \kappa$ is a strong limit cardinal in V;
- $M \models |\alpha| = \kappa$, for all $\alpha \in (\kappa, \lambda)$;
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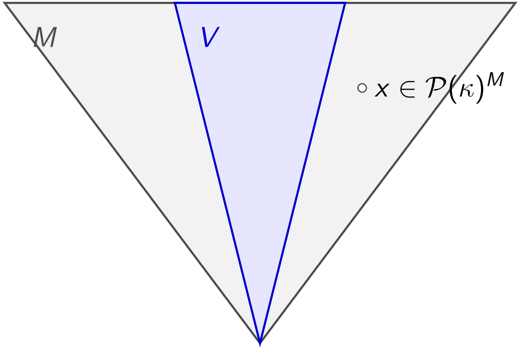
Then there is a $\mathbb{P}$-generic h in M.

M has the κ-trace property and collapses everything between κ and a strong limit λ.



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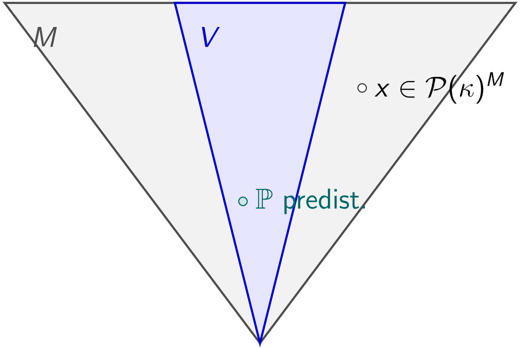

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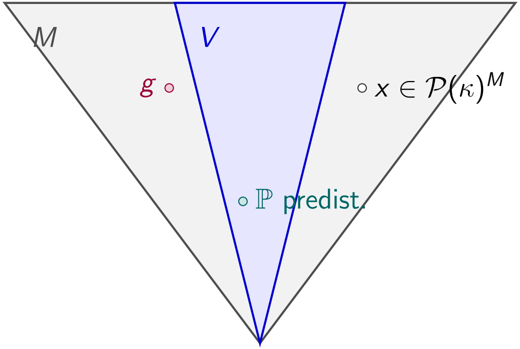

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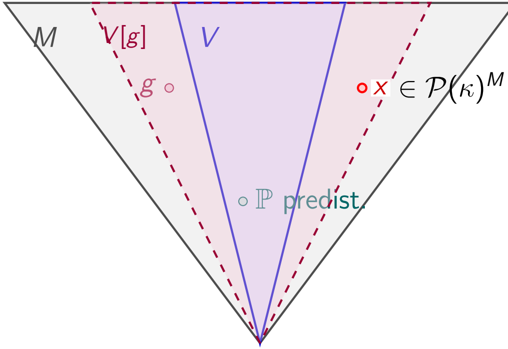

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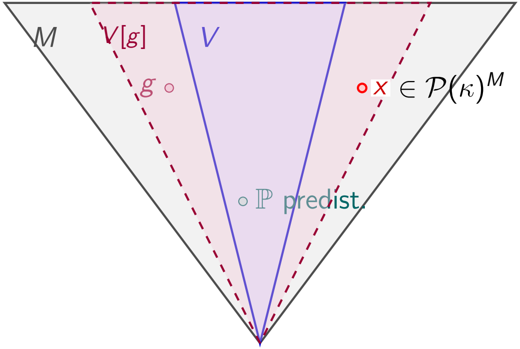

Definition

Let $\mathcal{P}(\kappa)^*$ be the set of all $x \in \mathcal{P}(\kappa)^M$ such that there is a (ω, κ)-predistributive $\mathbb{P}$ with $|\mathbb{P}|^V < \lambda$, and a $\mathbb{P}$-generic $g \in M$ with $x \in V[g]$.

$x \in \mathcal{P}(\kappa)^* :\iff x \in \mathcal{P}(\kappa)^M$ and there are an (ω, κ)-predistributive $\mathbb{P}$ and a $\mathbb{P}$-generic $g \in M$ such that $|\mathbb{P}|^V < \lambda$ and $x \in V[g]$

Theorem (T.)

Let κ be a λ-supercompact cardinal, where $\lambda > \kappa$ is inaccessible.

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- (•) Let $\mathbb{P}_{\mathcal{U}}$ denote the supercompact Prikry forcing.
- (•) Let $\mathrm{Mer}(\kappa, < \lambda)$ denote the Merimovich forcing.

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- (•) Let $\mathrm{Mer}(\kappa, < \lambda)$ denote the Merimovich forcing.

Suppose $M \supseteq V$ has the κ-trace property and $|\alpha|^M = \kappa$, for all $\alpha \in (\kappa, \lambda)$.

$x \in \mathcal{P}(\kappa)^* :\iff x \in \mathcal{P}(\kappa)^M$ and there are an (ω, κ)-predistributive $\mathbb{P}$ and a $\mathbb{P}$-generic $g \in M$ such that $|\mathbb{P}|^V < \lambda$ and $x \in V[g]$

Theorem (T.)

Let κ be a λ-supercompact cardinal, where $\lambda > \kappa$ is inaccessible.

- Let $\mathbb{P}_{\mathcal{U}}$ denote the supercompact Prikry forcing.
- Let $\mathrm{Mer}(\kappa, < \lambda)$ denote the Merimovich forcing.

Suppose $M \supseteq V$ has the κ-trace property and $|\alpha|^M = \kappa$, for all $\alpha \in (\kappa, \lambda)$. Then there exist a $\mathbb{P}_{\mathcal{U}}$-generic G and a $\mathrm{Mer}(\kappa, < \lambda)$-generic H such that

$$L(\mathcal{P}(\kappa)^*) = L(\mathcal{P}(\kappa)^*_G) = L(\mathcal{P}(\kappa)^{V[H]}),$$

where $\mathcal{P}(\kappa)^*_G := \bigcup_{\alpha \in (\kappa, \lambda)} \mathcal{P}(\kappa)^{V[G \restriction \alpha]}$.

$x \in \mathcal{P}(\kappa)^*_{M_i} :\iff x \in \mathcal{P}(\kappa)^{M_i}$ and there are an (ω, κ)-predistributive $\mathbb{P}$ and a $\mathbb{P}$-generic $g \in M_i$ such that $|\mathbb{P}|^V < \lambda$ and $x \in V[g]$

Corollary

Let κ be a λ-supercompact cardinal, where $\lambda > \kappa$ is inaccessible. Let $M_0 \supseteq V$ and $M_1 \supseteq V$ be models with the κ-trace property such that $|\alpha|^{M_0} = |\alpha|^{M_1} = \kappa$, for all $\alpha \in (\kappa, \lambda)$, $\mathcal{P}(\kappa)^*_{M_0} \subseteq \mathcal{P}(\kappa)^*_{M_1}$ and $(\kappa^+)^{M_0} = (\kappa^+)^{M_1}$.

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$$j : L(\mathcal{P}(\kappa)^*_{M_0}) \to L(\mathcal{P}(\kappa)^*_{M_1})$$

that fixes the ordinals.

Corollary

Let κ be a λ-supercompact cardinal, where $\lambda > \kappa$ is inaccessible. Suppose $M \supseteq V$ has the κ-trace property and $|\alpha|^M = \kappa$, for all $\alpha \in (\kappa, \lambda)$. Then

1. $\mathcal{P}(\kappa)^{L(\mathcal{P}(\kappa)^*)} = \mathcal{P}(\kappa)^*$;
2. $L(\mathcal{P}(\kappa)^*) \models \mathsf{DC}_\kappa$;
3. $L(\mathcal{P}(\kappa)^*) \models$ "$\mathrm{cof}(\kappa) = \omega \wedge \lambda = \kappa^+$";

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5. Approachability property AP_κ fails in $L(\mathcal{P}(\kappa)^*)$.

Questions

Suppose $M \supseteq V$ has the κ-trace property and collapses everything between κ and some strong limit λ.

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Questions

Suppose $M \supseteq V$ has the κ-trace property and collapses everything between κ and some strong limit λ.

(•) Can κ^+ be measurable in $L(\mathcal{P}(\kappa)^*)$?

(•) Is "κ^+ is $\mathcal{P}(\kappa)$-supercompact" consistent?

(•) Suppose κ has countable cofinality in V. Can there be an elementary embedding

$$j : L(\mathcal{P}(\kappa))^V \to L(\mathcal{P}(\kappa)^*)?$$

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THANK YOU

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