A pcf view on the HOD hypothesis

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If an inner model M is sufficiently *close* to V, then some of the properties of M may lift to V.

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for a proper class of singular cardinals which are singular in M.

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In V there is a measurable cardinal

$\Downarrow$

In M there is an inner model with a measurable cardinal

$(\lambda^+)^M = \lambda^+$ for a proper class of singular cardinals which are singular in M

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M has an inner model of a LC Axiom that holds in V

Jensen's L dichotomy

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2. Every uncountable cardinal is inaccessible in L.

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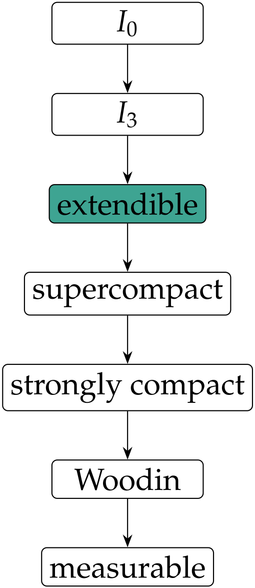
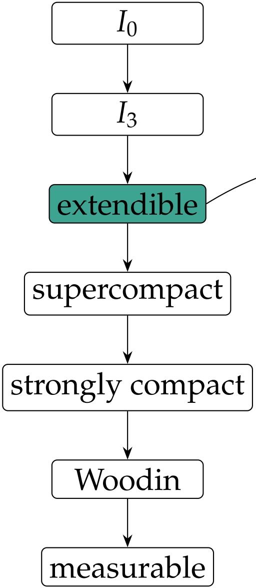


I_0
I_3
extendible
supercompact
strongly compact
Woodin
measurable



A cardinal κ is **extendible** if for all $\gamma > \kappa$ there is an elementary embedding

$$j\colon V_\gamma \to V_\theta$$

with

- $\mathrm{crit}(j) = \kappa$;
- $j(\kappa) > \gamma$.

ω-strong measurability

An uncountable regular cardinal μ is **ω-strongly measurable in HOD** if there is $\gamma < \mu$ such that

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1. γ is an infinite cardinal in HOD and $(2^\gamma)^{\mathrm{HOD}} < \mu$;
2. There is no partition

$$\langle S_\alpha \mid \alpha < \gamma \rangle \in \mathrm{HOD}$$

of $S^\mu_\omega := \{\eta < \mu \mid \mathrm{cof}(\eta) = \omega\}$ into stationary sets.

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If κ is extendible, exactly one of the following holds:

1. Every singular cardinal $\lambda > \kappa$ is singular in HOD and $(\lambda^+)^{\mathrm{HOD}} = \lambda^+$.
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Woodin's HOD dichotomy

If κ is extendible, exactly one of the following holds:

1. For every regular cardinal $\mu \geq \kappa$, HOD has the μ-cover property.
2. Every regular cardinal $\delta \geq \kappa$ is ω-strongly measurable in HOD.

The cover property

Let M be an inner model and let $\lambda > \omega$ be a cardinal.

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Let M be an inner model and let $\lambda > \omega$ be a cardinal.
M has the **λ-cover property** if for all $X \subseteq \mathrm{Ord}$ with $|X| < \lambda$, there is $Y \in M$ with $|Y| < \lambda$ such that $X \subseteq Y$.

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If κ is extendible, exactly one of the following holds:

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Conclusion (1) is known as the HOD **hypothesis**.

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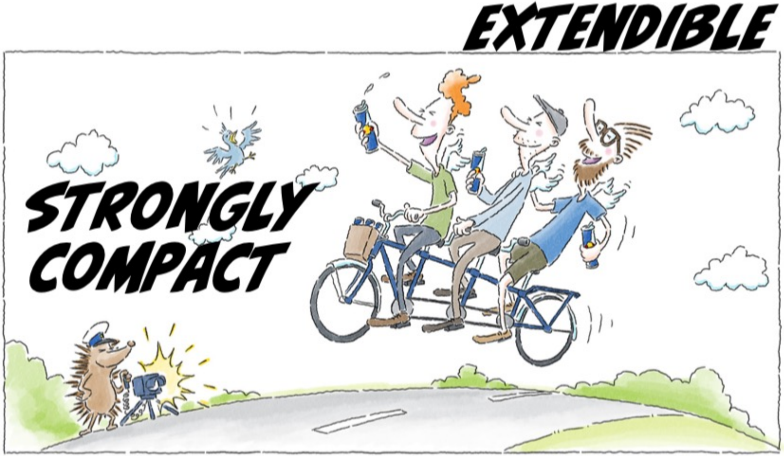
The HOD hypothesis

There is a proper class of regular cardinals that are not ω-strongly measurable in HOD.

Goldberg showed that the HOD dichotomy takes hold far below the least extendible cardinal:



Goldberg's HOD dichotomy

If κ is strongly compact, exactly one of the following holds:

1. HOD has the κ-cover property.
2. All sufficiently large regular cardinals are ω-strongly measurable in HOD.

HOD is able to cover small sets.

Is HOD able to cover big sets?

How close is HOD to V?

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Does the HOD hypothesis imply that HOD has the cover property above the first strongly compact cardinal?

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Proposition (Goldberg)

If κ is strongly compact and HOD has the κ-cover property, then

$$\text{HOD has the } \lambda\text{-cover property,}$$

for all strong limit cardinals $\lambda > \kappa$.

Towards an answer…

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Theorem (Magidor)

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1. M is cofinal correct.
2. $M \models$ GCH.
3. M has the ω_1-cover property.

Then M has the cover property above ω_1.

The proof of Magidor's theorem relies on an important piece of **pcf theory**; namely, the determination of conditions that ensure the existence of **exact upper bounds**.

PCF THEORY®

Scale

Let $A \subseteq \mathrm{Ord}$ be an infinite set and let $\alpha \in \mathrm{Ord}$ be uncountable. A **scale of length α on A** is a sequence of functions

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Shelah's representation theorem

Under AC, every singular cardinal λ of uncountable cofinality admits a scale of length λ^+ on $C^{(+)}$, for some club $C \subseteq \lambda$.

Theorem (Dimonte-Poveda-T., Straffelini-T.)

Assume that λ is supercompact. Then there is a model of $\mathrm{ZF} + \mathrm{DC}_\lambda + \neg\mathrm{AC}$ where:

- λ is a strong limit singular cardinal with $\mathrm{cof}(\lambda) = \omega$.
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Henceforth, M is an inner model of ZFC and κ is an uncountable regular cardinal

$$M \models \text{“}\vec{f}\text{ is a scale”} \;\leftrightsquigarrow\; \vec{f}\text{ is an }M\text{-scale}$$

$$M \models \text{“}\vec{f} \text{ is a scale”} \iff \vec{f} \text{ is an } M\text{-scale}$$

The scale property

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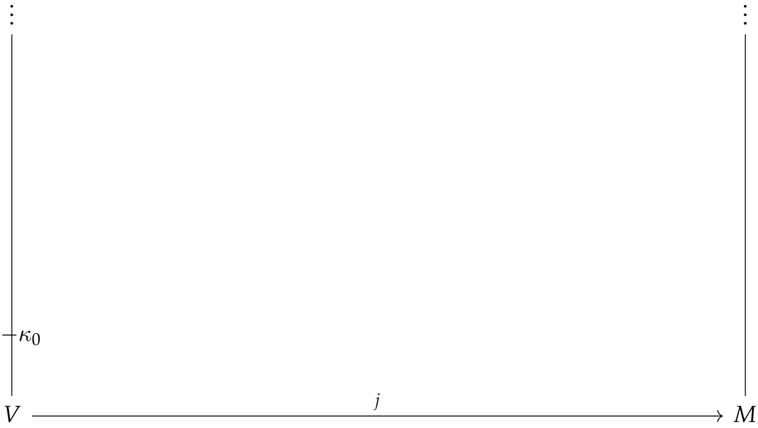

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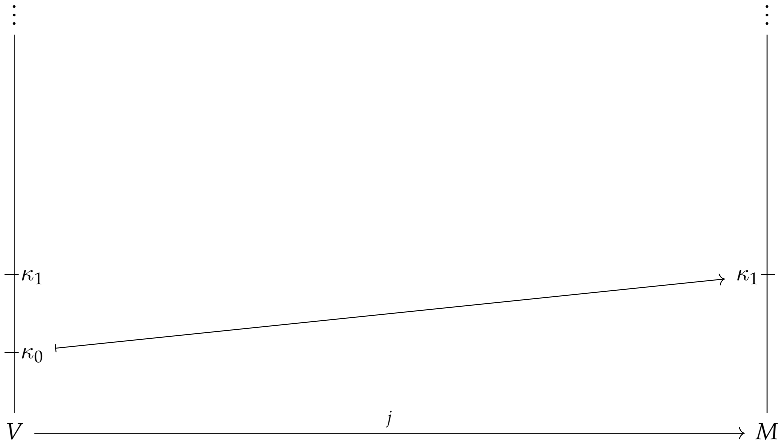
Let $\kappa_0 = \mathrm{crit}(j)$ and $\kappa_1 = j(\kappa_0)$.



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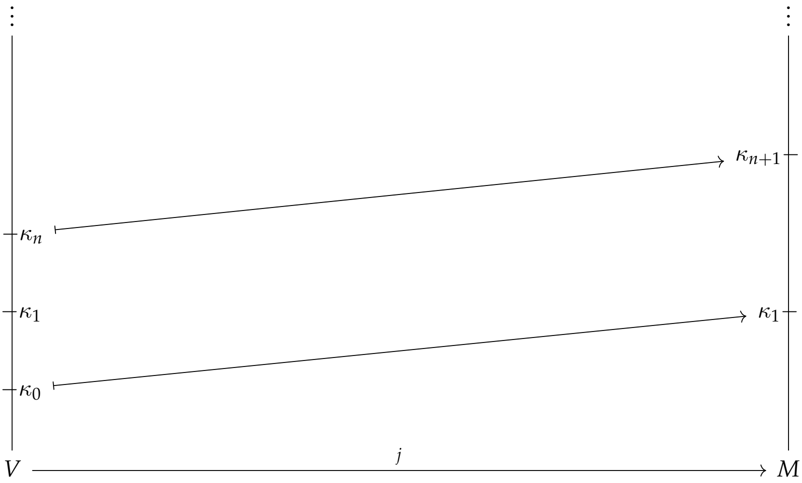
For $n \geq 1$, let $\kappa_{n+1} = j(\kappa_n)$.



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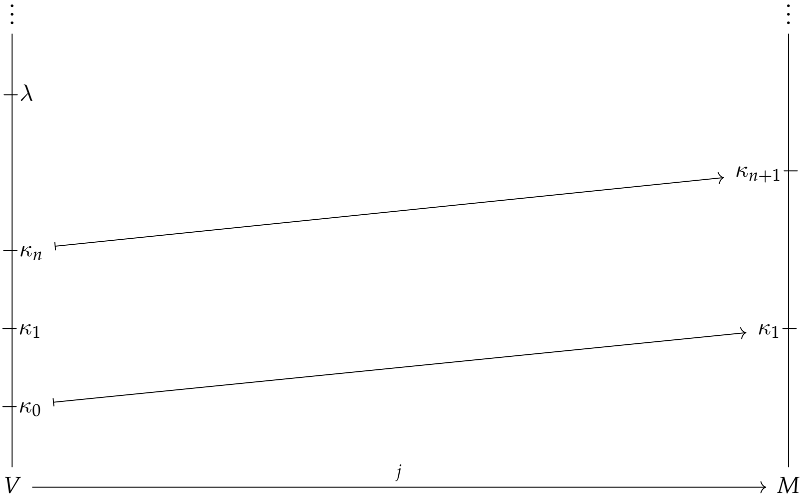
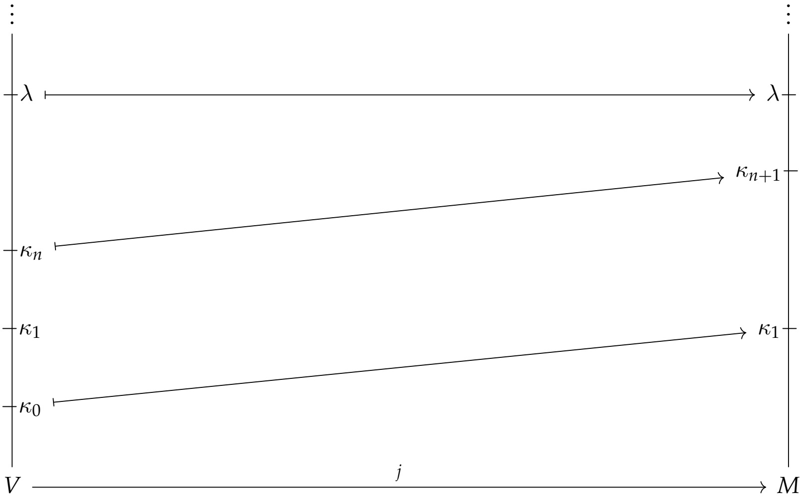


A failure of the scale property

I_2: There is an elementary embedding $j\colon V \to M$ with $V_\lambda \subseteq M$. Also, $j(\lambda) = j(\sup_{n<\omega} \kappa_n) = \sup_{n<\omega} j(\kappa_n) = \sup_{n<\omega} \kappa_{n+1} = \lambda$.



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Claim. $\langle g_\beta \mid \beta < \lambda^+ \rangle$ cannot be a scale.

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Claim. M does not have the λ-scale property.

Proof. Suppose, towards a contradiction, that there is an unbounded $C \subseteq j``D$ such that

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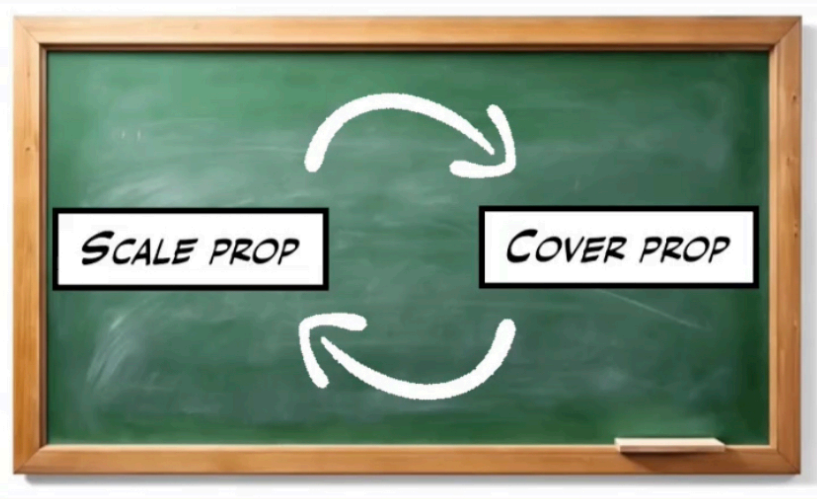
Then there is $\alpha < \lambda^+$ such that $h <^* g_{j(\alpha)} \restriction C = j(f_\alpha) \restriction C$. Now pick a big enough ν, with $j(\nu) \in C$, to get the following contradiction:

$$h(j(\nu)) < j(f_\alpha)(j(\nu)) = j(f_\alpha(\nu)) < \sup j``\nu = h(j(\nu)).$$

□


Scale prop
Cover prop

Theorem (T.)

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Then the following are equivalent:

1. M has the cover property above κ.
2. M has the λ-scale property, for all M-singular cardinals $\lambda > \kappa$ with $\mathrm{cof}(\lambda) < |\lambda|$.

Corollary (T.)

Assume the HOD hypothesis holds, and let κ be HOD-supercompact.

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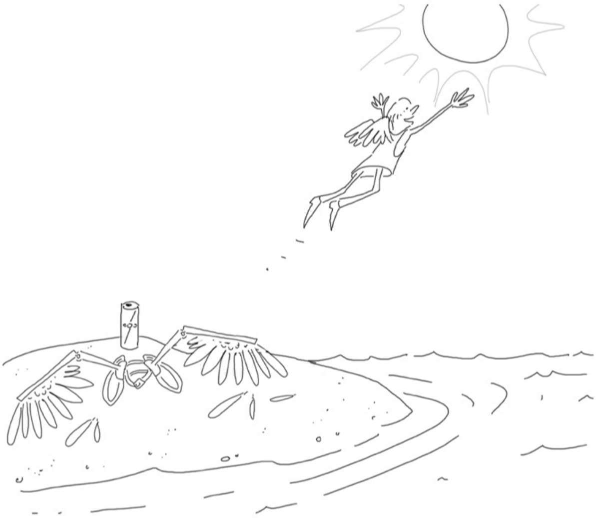
for all HOD-singular cardinals $\lambda > \kappa$ with $\mathrm{cof}(\lambda) < |\lambda|$.

THE SCALE PROPERTY



Proposition

If M has the κ-cover property and λ is an M-singular cardinal with $\mathrm{cof}(\lambda) < \kappa < \lambda$, then

$$M \text{ has the } \lambda\text{-scale property.}$$

M has the κ-cover property

$\lambda \in \mathrm{Sing}^M$	λ-scale property
$\mathrm{cof}(\lambda) < \kappa < \lambda$	✓

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$\mathrm{cof}(\lambda) < \kappa < \lambda$	✓
$\kappa \leq \mathrm{cof}(\lambda)$ and $\lvert\lambda\rvert$ is singular	
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$\lambda \in \text{Sing}^M$	λ-scale property
$\text{cof}(\lambda) < \kappa < \lambda$	✓
$\kappa \leq \text{cof}(\lambda)$ and $\lvert\lambda\rvert$ is singular	?
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M has the **intermediate cover property above κ**

$\lambda \in \mathrm{Sing}^M$	λ-scale property
$\mathrm{cof}(\lambda) < \kappa < \lambda$	✓
$\kappa \leq \mathrm{cof}(\lambda)$ and $\lvert\lambda\rvert$ is singular	✓
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Lemma (Goldberg)

The HOD hypothesis implies that for all HOD-regular cardinals δ above the first strongly compact cardinal,

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Cover prop. $\Longrightarrow$ Intermediate cover prop. $\Longrightarrow$ Weak cover prop.

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Theorem (T.)

If λ is an M-singular cardinal with $\kappa \leq \mathrm{cof}(\lambda)$ and $|\lambda| \in \mathrm{Sing}$, then

$$M \text{ has the } \lambda\text{-scale property.}$$

$$(\kappa \leq \mathrm{cof}(\lambda) \ \wedge \ |\lambda| \in \mathrm{Sing}) \Longrightarrow \lambda\text{-scale prop.}$$

Proof. Let $\vec{f} = \langle f_\beta \mid \beta < (\lambda^+)^M \rangle$ be an M-scale on $D^{(+M)}$.

$(\kappa \leq \mathrm{cof}(\lambda) \ \wedge \ |\lambda| \in \mathrm{Sing}) \Longrightarrow \lambda$-scale prop.

Proof. Let $\vec{f} = \langle f_\beta \mid \beta < (\lambda^+)^M \rangle$ be an M-scale on $D^{(+M)}$.

<u>**Step 1:**</u> Construct in M an auxiliary $<^*$-increasing sequence

$$\vec{g} = \langle g_\beta \mid \beta < (\lambda^+)^M \rangle$$

that admits an **exact upper bound** $h \colon D^{(+M)} \to \lambda$,

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Step 2: There is a club $C \subseteq D$ such that

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$\square$

$\lambda \in \mathrm{Sing}^M$	λ-scale property
$\mathrm{cof}(\lambda) < \kappa$	✓
$\kappa \leq \mathrm{cof}(\lambda)$ and $\lvert\lambda\rvert$ is singular	✓
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To handle this last case, we return to Magidor's theorem:

$$\begin{array}{|l|}\hline (1)\ M \models \text{GCH} \\ (2)\ M \text{ has the } \omega_1\text{-cover property} \quad \Longrightarrow M \text{ has the cover property} \\ (3)\ M \text{ is cofinal correct} \\ \hline \end{array}$$

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where

$$\mathrm{cof}(\mathcal{P}_{<\delta}(\alpha), \subseteq) := \min\{|A| \mid A \text{ is } \subseteq\text{-cofinal in } \mathcal{P}_{<\delta}(\alpha)\}.$$

Fact

If κ is strongly compact, then $\triangle_\kappa$ holds.[1]

[1] We acknowledge Lambie-Hanson for communicating the proof .

$\triangle$ under large cardinals

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Under **HOD-supercompactness** and the HOD hypothesis, the triangle principle holds in HOD.

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HOD-supercompactness

A cardinal is κ is HOD-**supercompact** if for every $\alpha > \kappa$, there is an elementary embedding $j\colon V \to M$ such that

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In particular, $\text{HOD} \models \triangle_\kappa$.

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Suppose that

(א) $M \models \triangle_\kappa$.

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Then the following are equivalent:

1. M has the cover property above κ.
2. M has the λ-scale property, whenever $\lambda > \kappa$ is M-singular with $\kappa \leq \mathrm{cof}(\lambda) < |\lambda|$ and $|\lambda| \in \mathrm{Reg}$.

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HOD dichotomy

If κ is HOD-supercompact, exactly one of the following holds:

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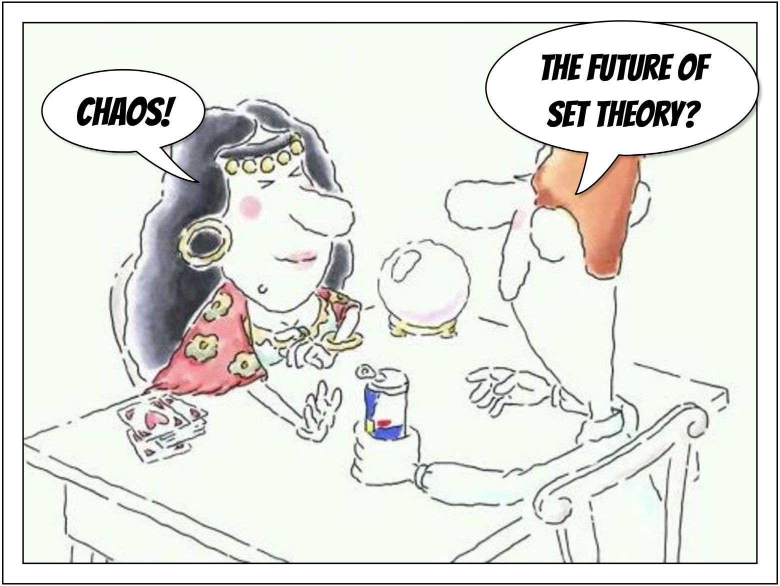
If κ is HOD-supercompact, exactly one of the following holds:

1. For every HOD-singular cardinal $\lambda > \kappa$ with $\mathrm{cof}(\lambda) < |\lambda|$, HOD has the λ-scale property.
2. Every regular cardinal $\delta \geq \kappa$ is ω-strongly measurable in HOD.

The failure of the HOD hypothesis



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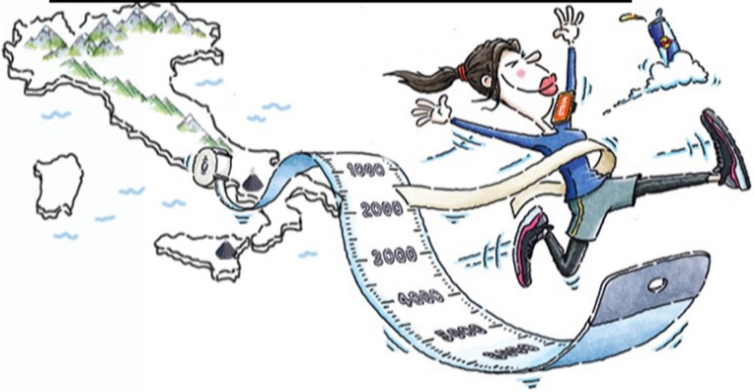
How far is HOD from V?

Does the failure of the HOD hypothesis imply the failure of the λ-scale property for all sufficiently large HOD-singular cardinals λ?

THANK YOU!



APPENDIX

Theorem (T.)

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(ב) M has the κ-cover property.

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Then the following are equivalent:

1. M has the cover property above κ.
2. M has the λ-scale property, whenever $\lambda > \kappa$ is M-singular with $\kappa \leq \mathrm{cof}(\lambda) < |\lambda|$ and $|\lambda| \in \mathrm{Reg}$.

$$\boxed{\left(\forall \lambda \in \mathrm{Sing}^M \, (\kappa \leq \mathrm{cof}(\lambda) < |\lambda| \ \wedge \ |\lambda| \in \mathrm{Reg}) \Rightarrow \text{cover prop.} > \kappa\right)}$$

Proof. By induction.

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Proof. By induction. Let $\delta > \kappa$ be a successor cardinal and let $X \in \mathcal{P}_{<\delta}(\lambda)$, for some ordinal $\lambda \geq \delta$.

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Nontrivial Case: λ is M-singular and $\mathrm{cof}(\lambda)^M \geq \kappa$.

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Nontrivial Case: λ is M-singular and $\mathrm{cof}(\lambda)^M \geq \kappa$.
Pick a club $D \in \mathcal{P}(\lambda)^M$ with $D \subseteq \mathrm{Sing}^M \setminus \delta$, and define

$$\langle \mathcal{C}_\alpha \mid \alpha \in D \rangle \in M$$

so that each $\mathcal{C}_\alpha$ is $\subseteq$-cofinal in $\mathcal{P}_{<\delta}(\alpha)^M$ and $|\mathcal{C}_\alpha|^M \leq (\alpha^+)^M$.

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$$d \colon (\alpha^+)^M \mapsto F_\alpha(Z_\alpha).$$

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By the λ-scale property there is $g \in M \cap \prod D^{(+M)}$ such that $d \restriction C^{(+M)} <^* g \restriction C^{(+M)}$, for some club $C \subseteq D$.

So for all sufficiently large $\alpha \in C$, we have

- $X \cap \alpha \subseteq Z_\alpha \in \mathcal{C}_\alpha \subseteq \mathcal{P}_{<\delta}(\alpha)^M$, and
- $d((\alpha^+)^M) = F_\alpha(Z_\alpha) < g((\alpha^+)^M) < (\alpha^+)^M$.

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Now pass to V and use Fodor's Theorem to find a stationary set $S \subseteq C$ on which the regressive function

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The set

$$\{h_\alpha(d((\alpha^+)^M)) \mid \alpha \in S\} \subseteq \beta$$

can be covered by some $Y \in M$ with $|Y| < \delta$.

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$$h_\alpha \colon g((\alpha^+)^M) \to \alpha.$$

Now pass to V and use Fodor's Theorem to find a stationary set $S \subseteq C$ on which the regressive function

$$\alpha \mapsto h_\alpha(d((\alpha^+)^M)),$$

defined on C, is bounded by some $\beta < \lambda$.
The set

$$\{h_\alpha(d((\alpha^+)^M)) \mid \alpha \in S\} \subseteq \beta$$

can be covered by some $Y \in M$ with $|Y| < \delta$. Finally, define

$$Z := \bigcup \{E \mid \exists \alpha \in D \text{ such that } E \in \mathcal{C}_\alpha \text{ and } h_\alpha(F_\alpha(E)) \in Y\}.$$

□